

RANK-2 SYZGY BUNDLES ON FERMAT CURVES AND AN APPLICATION TO HILBERT-KUNZ FUNCTIONS

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ABSTRACT. In this paper we describe the Frobenius pull-backs of the syzygy bundles $\mathrm{Syz}_C(X^a, Y^a, Z^a)$, $a \geq 1$, on the projective Fermat curve C of degree n in characteristics coprime to n , either by giving their strong Harder-Narasimhan filtration if $\mathrm{Syz}_C(X^a, Y^a, Z^a)$ is not strongly semistable or in the strongly semistable case by their periodicity behavior. Moreover, we apply these results to Hilbert-Kunz functions, to find Frobenius periodicities of the restricted cotangent bundle $\Omega_{\mathbb{P}^2|C}$ of arbitrary length and a problem of Brenner regarding primes with strongly semistable reduction.

INTRODUCTION

In this paper we study rank-2 syzygy bundles for certain monomial ideals, namely the ideals (X^a, Y^a, Z^a) for $a \geq 1$ on a smooth projective Fermat curve $C := \mathrm{Proj}(k[X, Y, Z]/(X^n + Y^n + Z^n))$. We focus mainly on positive characteristic, where semistability is not preserved by the Frobenius morphism. The first example is the restricted cotangent bundle $\Omega_{\mathbb{P}^2|C}$ from the projective plane which can be identified with the syzygy bundle $\mathrm{Syz}_C(X, Y, Z)$. For this bundle semistability is well known (if $n \geq 2$) but no complete answer is known regarding strong semistability, i.e., for what characteristics and curve degrees is $F^{e*}(\Omega_{\mathbb{P}^2|C})$ semistable for all $e \geq 0$.

The bundles $\mathrm{Syz}_C(X^a, Y^a, Z^a)$ on Fermat curves exhibit important arithmetic properties of vector bundles in general. For instance, Brenner proved in [Bre05b] with this setup that there exists no restriction theorem for strong semistability of Bogomolov type. In [Bre05a] he used the syzygy bundle $\mathrm{Syz}_C(X^2, Y^2, Z^2)$ on the Fermat quintic to show that strong semistability is not an open property in arithmetic deformations which was conjectured by N. I. Shepherd-Barron. In [BK08] Brenner and the second author disproved a conjecture of Joshi, which is related to the Grothendieck p -curvature conjecture, exploiting the arithmetic properties of Fermat curves and syzygy bundles. Despite these papers, there has been no complete treatment of the semistability properties of the bundles $\mathrm{Syz}_C(X^a, Y^a, Z^a)$ on Fermat curves.

Fermat rings are also popular examples for characteristic p methods in commutative algebra, in particular *Hilbert-Kunz theory* and the theory of *tight closure*. Monsky and Han exploited Fermat rings to obtain explicit computations for the Hilbert-Kunz multiplicity and the Hilbert-Kunz function (see [HM93], [Han91], [Mon06b] and [Mon06a]), which we partially complement in this paper. Due to work of Brenner [Bre06] and Trivedi [Tri05] there is a strong relationship to strongly semistable vector bundles which is our main tool to compute Hilbert-Kunz functions in this

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paper. We remark that for the arithmetic behavior of tight closure the ideals (X^a, Y^a, Z^a) which are topic of this paper are useful examples (see [BK06], [Sin98]). In Section 1 we recall the necessary notations of Hilbert-Kunz theory and vector bundles and discuss their relationship mentioned above for syzygy bundles of rank 2. Afterwards, we turn our attention to *syzygy gaps* and how they appear in Han's and Monsky's computations [HM93] of Hilbert-Kunz multiplicities of ideals generated by a fixed (positive) power of the variables in two-dimensional Fermat rings.

Section 2 is devoted to the case where the bundle $\mathrm{Syz}_C(X^a, Y^a, Z^a)$, $a \geq 1$, is strongly semistable. We will show how work of Kustin, Rahmati and Vraciu [KRV12] can be applied in this situation to obtain a periodicity up to twist of the Frobenius pull-backs of these bundles as well as a sharp bound for the length of this periodicity (cf. Theorem 2.9). This periodic behavior enables us to compute the Hilbert-Kunz functions of the ideals (X^a, Y^a, Z^a) , $a \geq 1$, in the Fermat rings $k[X, Y, Z]/(X^n + Y^n + Z^n)$ (cf. Theorem 2.12) and allows us to generalize a theorem of Brenner and the second author (cf. Theorem 2.14).

In Section 3 we will use the methods developed in Section 2 to construct for every odd prime p and every natural number l an n such that the Frobenius pull-backs of $\Omega_{\mathbb{P}^2|C}$ admit up to twist a periodicity of length l , where C is the projective Fermat curve of degree n (see Example 4.1). Moreover, we will see in Example 4.2 that in characteristic two, the Frobenius pull-backs of the restricted cotangent bundle admit a periodicity up to twist if and only if $n = 3$.

In Section 4 we investigate the case where $\mathrm{Syz}_C(X^a, Y^a, Z^a)$ is not strongly semistable. In this case we can explicitly describe the minimal e_0 such that the e_0 -th Frobenius pull-back of this bundle has a strong Harder-Narasimhan filtration. Moreover, this filtration is explicitly computed (cf. Theorem 4.4). This result is used to complete the computations of Hilbert-Kunz functions from Section 2 (cf. Corollary 4.7). The explicit computation of the entire Hilbert-Kunz function for ideals (X^a, Y^a, Z^a) with $a > 1$ is new (at least to the best knowledge of the authors of this paper).

Finally, in Section 5 we deal with a special instance of the Miyaoka problem proposed by Brenner, i.e., the question how the property of $\mathcal{S} := \mathrm{Syz}_C(X^a, Y^a, Z^a)$ being strongly semistable depends on the parameter a , the characteristic p and the degree n of the projective Fermat curve C . Theorem 5.2 will show that \mathcal{S} is semistable in characteristic zero if and only if it is strongly semistable in all characteristics $p \equiv \pm 1$ modulo $2n$. From this result we will deduce that the Harder-Narasimhan filtration of \mathcal{S} in characteristic zero can be computed via reduction modulo p (cf. Theorem 5.5).

We remark that most of the results in this paper can easily be translated into algorithms that are suitable for a computer algebra system (e.g., [CoC]).

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1. PRELIMINARIES

Let (R, \mathfrak{m}) be a local Noetherian ring of characteristic $p > 0$ and dimension d . Let $I = (f_1, \dots, f_m)$ be an \mathfrak{m} -primary ideal. Let λ_R denote the length function for

R -modules. The function

$$\mathrm{HK}(I, p^e) : \mathbb{N} \longrightarrow \mathbb{N}, \quad e \longmapsto \lambda_R \left(R / \left(f_1^{p^e}, \dots, f_m^{p^e} \right) \right)$$

is called the *Hilbert-Kunz function* of I . We call the limit

$$\lim_{e \rightarrow \infty} \frac{\mathrm{HK}(I, p^e)}{p^{ed}},$$

whose existence was proven by Monsky in [Mon83], the *Hilbert-Kunz multiplicity* of I . We denote this limit by $e_{\mathrm{HK}}(I)$ and call $e_{\mathrm{HK}}(R) := e_{\mathrm{HK}}(\mathfrak{m})$ the Hilbert-Kunz multiplicity of R . Sometimes we will use the symbol $(f_1, \dots, f_m)^{[p^e]}$ to denote the ideal $(f_1^{p^e}, \dots, f_m^{p^e})$.

Throughout this paper we are mainly interested in the graded situation, where k is an algebraically closed field and R a standard-graded normal k -algebra of dimension 2. Moreover, we consider homogeneous elements $f_1, f_2, f_3 \in R$ with $\deg(f_1) = \deg(f_2) = \deg(f_3) = a \geq 1$ such that the ideal $I := (f_1, f_2, f_3) \subset R$ is R_+ -primary. These elements give rise to the short exact (presenting) sequences

$$0 \longrightarrow \mathrm{Syz}_C(f_1, f_2, f_3)(m) \longrightarrow \bigoplus_{i=1}^3 \mathcal{O}_C(m-a) \xrightarrow{f_1, f_2, f_3} \mathcal{O}_C(m) \longrightarrow 0,$$

for all $m \in \mathbb{Z}$ on the smooth projective curve $C := \mathrm{Proj}(R)$. The kernel sheaf $\mathrm{Syz}_C(f_1, f_2, f_3)$ is locally free of rank 2 and is called the *syzygy bundle* for f_1, f_2, f_3 . In the case $\mathrm{char}(k) = p > 0$ the Hilbert-Kunz function of the ideal I can be computed via the formula

$$(1.1) \quad \dim_k \left(R / \left(f_1^{p^e}, f_2^{p^e}, f_3^{p^e} \right) \right)_m = h^0(C, \mathcal{O}_C(m)) - \sum_{i=1}^3 h^0(C, \mathcal{O}_C(m - p^e a)) \\ + h^0 \left(C, \mathrm{Syz}_C \left(f_1^{p^e}, f_2^{p^e}, f_3^{p^e} \right) (m) \right)$$

and summation over all $m \in \mathbb{Z}$ (see for instance [Bre06]). If \mathcal{S} is a vector bundle on a smooth projective curve over an algebraically closed field k we define its *slope* by the quotient $\mu(\mathcal{S}) := \frac{\deg(\mathcal{S})}{\mathrm{rank}(\mathcal{S})}$ with $\deg(\mathcal{S}) := \deg(\bigwedge^{\mathrm{rank} \mathcal{S}} \mathcal{S})$. We recall that \mathcal{E} is *semistable* if for every non-trivial locally free subsheaf \mathcal{F} the inequality $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ holds. The bundle \mathcal{E} is *stable* if this inequality is always strict. Hence the rank-2 syzygy bundle $\mathcal{S} := \mathrm{Syz}_C(f_1, f_2, f_3)$ is not semistable if and only if there exists a line bundle $0 \neq \mathcal{L} \subset \mathcal{E}$ with $\deg(\mathcal{L}) > \mu(\mathcal{S}) = \deg(\mathcal{S})/2 = -3an/2$, where $n = \deg(C)$. For such a line bundle of maximal degree this filtration constitutes the so-called *Harder-Narasimhan filtration* (or *HN-filtration*) of \mathcal{S} and the line bundle \mathcal{L} is the *(maximal) destabilizing subbundle*. We often write the HN-filtration of a rank-2 vector bundle as a short exact sequence $0 \rightarrow \mathcal{L} \rightarrow \mathcal{S} \rightarrow \mathcal{M} \rightarrow 0$, where the quotient \mathcal{M} is a line bundle with $\deg(\mathcal{M}) < \deg(\mathcal{L})$. Note that concept and existence of a HN-filtration is trivial for bundles of rank 2 but for bundles of higher rank it is much more complicated (see [HN75]).

In positive characteristic p we consider the *absolute Frobenius morphism* $F : C \rightarrow C$ which is the identity on the curve C and the p -th power map on the structure sheaf \mathcal{O}_C . It is well-known that the Frobenius pull-back $F^*(\mathcal{S})$ of a semistable vector bundle \mathcal{S} is in general not semistable (see for instance [Har71, Example 3.2] for Serre's counter example). If $F^{e*}(\mathcal{S})$ is semistable for all $e \geq 0$ then \mathcal{S} is called *strongly semistable*. This notion is due to Miyaoka (cf. [Miy87, Section 5]). In the

case of a rank-2 vector bundle the HN-filtration $0 \rightarrow \mathcal{L} \rightarrow F^{e*}(\mathcal{S}) \rightarrow \mathcal{M} \rightarrow 0$ of a (non-semistable) Frobenius pull-back $F^{e*}(\mathcal{S})$ is called the *strong HN-filtration* of the bundle \mathcal{S} . We indicate that, as for the HN-filtration itself, the concept and existence of a strong HN-filtration is highly non-trivial for vector bundles of higher rank (see [Lan04] for a detailed account).

Due to the work of Brenner [Bre06] and Trivedi [Tri05] strongly semistable vector bundles are closely related to Hilbert-Kunz theory. We only state this connection for the rank-2 vector bundles which we consider in this paper.

Theorem 1.1. *Let $C = V_+(G) \subset \mathbb{P}^2$ denote a smooth plane curve of degree n over an algebraically closed field k of positive characteristic p . Let $R = k[X, Y, Z]/(G)$ be its homogeneous coordinate ring and let $I = (f_1, f_2, f_3)$ denote a homogeneous R_+ -primary ideal such that $\deg(f_1) = \deg(f_2) = \deg(f_3) = a$. Let $\mathcal{S} := \text{Syz}_C(f_1, f_2, f_3)$. Then the following hold.*

- (1) *The bundle \mathcal{S} is strongly semistable if and only if $e_{\text{HK}}(I) = \frac{3n}{4}a^2$.*
- (2) *If \mathcal{S} is not strongly semistable with strong Harder-Narasimhan filtration $0 \rightarrow \mathcal{L} \rightarrow F^{e*}(\mathcal{S}) \rightarrow \mathcal{M} \rightarrow 0$, then*

$$e_{\text{HK}}(I) = n \left(\left(\frac{\deg(\mathcal{L})}{np^e} + \frac{3a}{2} \right)^2 + \frac{3a^2}{4} \right).$$

Moreover, we have

$$\deg(\mathcal{L}) = -\frac{3anp^e}{2} + np^e \cdot \sqrt{-\frac{3a^2}{4} + \frac{e_{\text{HK}}(I)}{n}} \in \left(-\frac{3anp^e}{2}, -anp^e \right].$$

- (3) *If \mathcal{S} is not semistable, then the Hilbert-Kunz multiplicity of I equals*

$$e_{\text{HK}}(I) = \frac{3n}{4}a^2 + \frac{\ell^2}{4n},$$

where $0 < \ell \leq na$ is an integer with $\ell \equiv an \pmod{2}$.

- (4) *If the bundle \mathcal{S} is semistable, but not strongly semistable, then the Hilbert-Kunz multiplicity of I equals*

$$e_{\text{HK}}(I) = \frac{3n}{4}a^2 + \frac{\ell^2}{4np^{2e}},$$

where $e \geq 1$ is the number such that $F^{e-1}(\mathcal{S})$ is semistable and $F^{e*}(\mathcal{S})$ is not semistable and $0 < \ell \leq n(n-3)$ is an integer with $\ell \equiv pna \pmod{2}$.*

Proof. For the proof of (1) and (2) see [Bre06, proof of Corollary 4.6]. For the proof of (3) and (4) see [Tri05, proof of Theorem 5.3]. In all three cases the quoted proofs only consider the case $a = 1$ but they are easy to generalize. For an explicit proof see [Kai09, proofs of Corollary 1.4.9 and Proposition 1.4.11]. \square

Next, we discuss syzygy gaps and how they appear in the computation of Hilbert-Kunz multiplicities. The final result of Han and Monsky (cf. Theorem 1.4) will combine the case by case description of the Hilbert-Kunz multiplicities of the ideals (X^a, Y^a, Z^a) from the last theorem. Moreover, Theorem 1.4 will give us a numerical criterion to check the strong semistability of the sheafs $\text{Syz}_C(X^a, Y^a, Z^a)$.

Definition 1.2. Let $R := k[X, Y]$ and let $f_1, f_2, f_3 \in R$ be non-zero and homogeneous. By Hilbert's Syzygy Theorem we have a splitting $\text{Syz}_R(f_1, f_2, f_3) \cong R(-\alpha) \oplus R(-\beta)$ for some $\beta \leq \alpha \in \mathbb{N}$. We call the difference $\alpha - \beta$ the *syzygy gap* of

f_1, f_2, f_3 and denote it by $\text{syzgap}(f_1, f_2, f_3)$. In the special case $f_1 = X^a, f_2 = Y^b, f_3 = (X+Y)^c$ for some positive integers a, b, c , we denote $\text{syzgap}(X^a, Y^b, (X+Y)^c)$ by $\delta(a, b, c)$.

One can show that δ extends to a unique continuous function δ^* on $[0, \infty)^3$ with the property $|\delta(t) - \delta(s)| \leq \|t - s\|_1$. Moreover, if the underlying polynomial ring is defined over a field of positive characteristic p , the extension of δ satisfies $\delta^*\left(\frac{t}{p}\right) = p^{-1} \cdot \delta^*(t)$. In what follows we will not distinguish between δ and δ^* . Our next goal is to explain how δ can be computed. The reference is Han's thesis [Han91] resp. [Mon06b] for an alternative proof.

We are interested in the *taxicab distance* of elements of the form $\frac{t}{p^s}$ to the set

$$L_{\text{odd}} := \left\{ u \in \mathbb{N}^3 \mid u_1 + u_2 + u_3 \text{ is odd} \right\},$$

where s is an integer and t a three-tuple of non-negative real numbers. Note that for given $\frac{t}{p^s}$ there is at most one $u \in L_{\text{odd}}$ satisfying $\left\| \frac{t}{p^s} - u \right\|_1 < 1$. Moreover, the only candidates for u_i are the rounding ups and rounding downs of $\frac{t_i}{p^s}$.

Theorem 1.3 (Han). *Let $t = (t_1, t_2, t_3) \in [0, \infty)^3$. If the t_i do not satisfy the strict triangle inequality (w.l.o.g. $t_1 \geq t_2 + t_3$), we have $\delta(t) = t_1 - t_2 - t_3$. If the t_i satisfy the strict triangle inequality and there are $s \in \mathbb{Z}$, $u \in L_{\text{odd}}$ with $\|p^s t - u\|_1 < 1$, then there is such a pair (s, u) with minimal s and with this pair (s, u) we get*

$$\delta(t) = \frac{1}{p^s} \cdot (1 - \|p^s t - u\|_1).$$

Otherwise, one has $\delta(t) = 0$.

With the help of the δ -function we can state the following theorem.

Theorem 1.4 (Han, Monsky). *The Hilbert-Kunz multiplicity of an ideal $I := (X^a, Y^a, Z^a)$, $a \geq 1$, of the Fermat ring $k[X, Y, Z]/(X^n + Y^n + Z^n)$ equals*

$$\frac{3a^2n}{4} + \frac{n^3}{4} \cdot \delta\left(\frac{a}{n}, \frac{a}{n}, \frac{a}{n}\right)^2.$$

Proof. The case $a = 1$ is due to Han (cf. [Han91, Theorem 2.30]) and was generalized by Monsky (cf. [Mon06a, Theorem 2.3]). \square

Combining Theorem 1.1(1) and Theorem 1.4 one obtains the following numerical criterion for strong semistability.

Corollary 1.5. *Let $\gcd(p, n) = 1$. The bundle $\text{Syz}_{V_+(X^n + Y^n + Z^n)}(X^a, Y^a, Z^a)$ is strongly semistable if and only if $\delta\left(\frac{a}{n}, \frac{a}{n}, \frac{a}{n}\right) = 0$.*

Example 1.6. In [Bre05a, Remark 2] Brenner asks whether the syzygy bundle $\text{Syz}_C(X^2, Y^2, Z^2)$ is strongly semistable on the Fermat curve C of degree n in characteristics $p \equiv \pm 1 \pmod{n}$. Using Corollary 1.5 we are able to answer this question positively for the interesting cases $n \geq 3$. Since $2p^s \equiv \pm 2 \pmod{2n}$ for every $s \geq 0$, we see that the distance of $\frac{2p^s}{n}$ to the next odd integer is $\frac{n-2}{n}$ and $3\frac{n-2}{n} = 3 - \frac{6}{n} \geq 1$. Thus, $\delta\left(\frac{2}{n}, \frac{2}{n}, \frac{2}{n}\right) = 0$ and $e_{\text{HK}}((X^2, Y^2, Z^2)) = \frac{12}{n}$, which is equivalent to the strong semistability of the syzygy bundle $\text{Syz}_C(X^2, Y^2, Z^2)$. In the case $n = 1$ we have $\text{Syz}_C(X^2, Y^2, Z^2) \cong \mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ and this bundle is obviously strongly semistable. For $n = 2$ the curve equation $X^2 + Y^2 + Z^2 = 0$ yields

a non-trivial global section of total degree 2. Since $\deg(\text{Syz}_C(X^2, Y^2, Z^2)(2)) = -4$, the bundle is not semistable.

2. THE STRONGLY SEMISTABLE CASE

We start by fixing the notations which will be used for the rest of this paper.

Situation 2.1. Let k be an algebraically closed field of characteristic $p > 0$ and let $R := k[X, Y, Z]/(X^n + Y^n + Z^n)$ for $n \geq 1$. We denote the projective Fermat curve of degree n by $C := \text{Proj}(R)$. We assume $\gcd(p, n) = 1$, hence C is a smooth curve. We will consider the ideal $I := (X^a, Y^a, Z^a)$, $a \geq 1$, as well as its first syzygy module $\text{Syz}_R(I)$ and the bundle $\text{Syz}_C(I)$.

An important special case in the situation above is $a = 1$, since $\text{Syz}_C(X, Y, Z)$ is isomorphic to the restricted cotangent bundle $\Omega_{\mathbb{P}^2|C}$ which can be seen by using the Euler sequence. Note that the restricted cotangent bundle $\Omega_{\mathbb{P}^2|C}$ is (at least) semistable on every smooth plane curve of degree ≥ 2 (cf. [Tri05, Corollary 3.5]). In this section we deal with the case where the syzygy bundle $\text{Syz}_C(X^a, Y^a, Z^a)$ is strongly semistable. For this purpose we will use results of [KRV12] where the authors study the minimal projective resolution of the quotients $R/(X^a, Y^a, Z^a)$ depending on a and n .

We start with the special case where some Frobenius pull-back of $\text{Syz}_C(X^a, Y^a, Z^a)$ becomes a direct sum of twisted copies of the structure sheaf. Hence we have an isomorphism

$$\text{Syz}_C(X^{aq}, Y^{aq}, Z^{aq}) \cong \mathcal{O}_C(l)^2$$

for some $q = p^e$ and some $l \in \mathbb{Z}$ due to the semistability of the pull-back. Note that this may happen only if $p = 2$ or if a is even as one sees by comparing the degrees of the bundles above.

By [Mil00, Theorem 2.1] the Hilbert-Kunz function is given by $e_{\text{HK}}(I^{[e_0]}) \cdot p^{2(e-e_0)}$, where e_0 is (minimal) such that $R/I^{[e_0]}$ has finite projective dimension.

The minimal e such that $\text{Syz}_C(X^{ap^e}, Y^{ap^e}, Z^{ap^e})$ is a direct sum of twisted copies of the structure sheaf can be computed with the following theorem.

Theorem 2.2 (Kustin, Rahmati, Vraciu). *In Situation 2.1, the quotient R/I has finite projective dimension as an R -module if and only if one of the following conditions is satisfied*

- (1) $n|a$,
- (2) $p = 2$ and $n \leq a$,
- (3) p is odd and there exist positive integers J and e with J odd such that

$$\left| Jp^e - \frac{a}{n} \right| < \begin{cases} 3^{e-1} & \text{if } p = 3, \\ \frac{p^e-1}{3} & \text{if } p^e \equiv 1 \pmod{3}, \\ \frac{p^e+1}{3} & \text{if } p^e \equiv 2 \pmod{3}. \end{cases}$$

Proof. See [KRV12, Theorem 6.3]. □

In [Li13] the author uses this theorem to study the interaction between strong semistability of $\text{Syz}_C(I)$, the projective dimensions of $R/I^{[q]}$ and the diagonal F-threshold of I .

In the case where $\text{Syz}_C(X^a, Y^a, Z^a)$ is strongly semistable but no Frobenius pull-back of it splits as a direct sum of twisted copies of the structure sheaf, we have that

all quotients $R/(X^{aq}, Y^{aq}, Z^{aq})$ have infinite projective dimension as R -module, because the first syzygy modules $\text{Syz}_R(X^{aq}, Y^{aq}, Z^{aq})$ are not free. In this case we can use a result of Kustin, Rahmati and Vraciu to obtain a twisted Frobenius periodicity of $\text{Syz}_C(X^a, Y^a, Z^a)$ in the following sense.

Definition 2.3. Let S be a normal, standard-graded k -domain of dimension two, where k is an algebraically closed field of characteristic $p > 0$. Let \mathcal{E} be a vector bundle over the smooth projective curve $Y := \text{Proj}(S)$. Assume there are $0 \leq s < t \in \mathbb{N}$ such that the Frobenius pull-backs $F^{e*}(\mathcal{E})$ of \mathcal{E} (and all their twists) are pairwise non-isomorphic for $0 \leq e \leq t-1$ and $F^{t*}(\mathcal{E}) \cong F^{s*}(\mathcal{E})(m)$ holds for some $m \in \mathbb{Z}$. We say that \mathcal{E} admits a *twisted (s, t) -Frobenius periodicity*. The bundle \mathcal{E} admits a *twisted Frobenius periodicity* if there are $0 \leq s < t \in \mathbb{N}$ such that \mathcal{E} admits a twisted (s, t) -Frobenius periodicity.

To state the necessary result of Kustin, et al., we need one more definition.

Definition 2.4. Let r and s be positive integers with $r + s = n$. Then we define

$$\phi_{r,s} := \begin{pmatrix} 0 & Z^r & -Y^r & X^s \\ -Z^r & 0 & X^r & Y^s \\ Y^r & -X^r & 0 & Z^s \\ -X^s & -Y^s & -Z^s & 0 \end{pmatrix}.$$

Theorem 2.5 (Kustin, Rahmati, Vraciu). *In Situation 2.1, let $a = \theta \cdot n + r$ with $\theta \in \mathbb{N}$ and $r \in \{1, \dots, n-1\}$. Assume that $Q := R/I$ has infinite projective dimension. If $\theta = 2 \cdot \eta - 1$, then the homogenous minimal free resolution of Q is given by*

$$\dots \xrightarrow{\phi_{r,n-r}} F_1(-n) \xrightarrow{\phi_{n-r,r}} F_2 \xrightarrow{\phi_{r,n-r}} F_1 \longrightarrow R(-a)^3 \longrightarrow R$$

with free graded modules

$$F_1 := R(-3\eta n + n - r)^3 \oplus R(-3\eta n + 2n - 3r),$$

$$F_2 := R(-3\eta n + n - 2r)^3 \oplus R(-3\eta n).$$

If $\theta = 2 \cdot \eta$, then the homogenous minimal free resolution of Q is given by

$$\dots \xrightarrow{\phi_{n-r,r}} F_1(-n) \xrightarrow{\phi_{r,n-r}} F_2 \xrightarrow{\phi_{n-r,r}} F_1 \longrightarrow R(-a)^3 \longrightarrow R,$$

where the first three free graded modules are defined as

$$F_1 := R(-3\eta n - 2r)^3 \oplus R(-3\eta n - n),$$

$$F_2 := R(-3\eta n - n - r)^3 \oplus R(-3\eta n - 3r)$$

Proof. The statement follows from [KRV12, Theorem 3.5] combined with [KRV12, Theorems 5.14 and 6.1]. \square

Corollary 2.6. *Under the hypothesis of Theorem 2.5, we have*

$$\begin{aligned} \text{Syz}_R(X^a, Y^a, Z^a)(m) &\cong \begin{cases} \text{coker}(\phi_{n-r,r}) & \text{if } \theta \text{ is even,} \\ \text{coker}(\phi_{r,n-r}) & \text{if } \theta \text{ is odd,} \end{cases} \\ &\cong \begin{cases} \text{Syz}_R(X^r, Y^r, Z^r) & \text{if } \theta \text{ is even,} \\ \text{Syz}_R(X^{n-r}, Y^{n-r}, Z^{n-r}) & \text{if } \theta \text{ is odd} \end{cases} \end{aligned}$$

for some $m \in \mathbb{Z}$.

Proof. The first isomorphism is clear from the free resolution. The second isomorphism follows from the free resolution by considering the case $a = r$. \square

At this point it is not clear whether the modules $M_r = \text{Syz}_R(X^r, Y^r, Z^r)$ for $1 \leq r \leq n-1$ are pairwise non-isomorphic or not. A criterion to decide this is given by the Hilbert-series, which can be computed with the help of the next theorem, which is a slight but useful improvement of [Bre05a, Lemma 1.1]. Geometrically spoken, it considers a smooth projective curve

$$D := V_+(Z^n - F(X, Y)) \subset \mathbb{P}^2 = \text{Proj } k[X, Y, Z],$$

where $F(X, Y) \in k[X, Y]$ denotes a homogeneous polynomial of degree n , and relates the sheaves $\text{Syz}_D(X^a, Y^b, Z^c)$ to the sheaves $\text{Syz}_D(X^a, Y^b, F(X, Y)^i)$ which come from \mathbb{P}^1 via the Noetherian normalization $D \rightarrow \mathbb{P}^1 = \text{Proj } k[X, Y]$. We will use the following result several times in the sequel of this paper.

Theorem 2.7. *Let k be a field, $S := k[X, Y, Z]/(Z^n - F(X, Y))$ and $F \in k[X, Y]$ homogeneous of degree $n \geq 2$. Let $a, b, c \geq 1$ and write $c = n \cdot q + r$ with $0 \leq r \leq n-1$ and $q \in \mathbb{N}$. For all $s \in \mathbb{Z}$ we have a short exact sequence*

$$\begin{aligned} 0 &\longrightarrow \text{Syz}_S(X^a, Y^b, Z^{c+n-2r})(s-r) \\ &\xrightarrow{\psi} \text{Syz}_S(X^a, Y^b, F^q)(s-r) \oplus \text{Syz}_S(X^a, Y^b, F^{q+1})(s) \\ &\xrightarrow{\phi} \text{Syz}_S(X^a, Y^b, Z^c)(s) \longrightarrow 0, \end{aligned}$$

where the maps are defined via

$$\begin{aligned} \psi(h_1, h_2, h_3) &:= ((Z^{n-r} \cdot h_1, h_2, h_3), (-h_1, -Z^r \cdot h_2, -Z^r \cdot h_3)) \\ \phi((f_1, f_2, f_3), (g_1, g_2, g_3)) &:= (f_1 + Z^{n-r} \cdot g_1, Z^r \cdot f_2 + g_2, Z^r \cdot f_3 + g_3). \end{aligned}$$

Proof. The injectivity of ψ is clear and the exactness at the middle spot is straightforward. The proof that ϕ is surjective can be found in [BK13, Lemma 2.1]. See also [Bri13, Chapter 4] for a detailed proof and a generalization. \square

Theorem 2.8. *The notations are the same as in Theorem 2.7. For $l \in \mathbb{N}$ we use the abbreviation $\mathcal{S}_l := \text{Syz}_S(X^a, Y^b, Z^l)$. Then the Hilbert-series of $\mathcal{S}_c = \text{Syz}_S(X^a, Y^b, Z^c)$ is given by*

$$H_{\mathcal{S}_c}(t) = \frac{(t^r - t^n) \cdot H_{\mathcal{S}_{c-r}}(t) + (1 - t^r) H_{\mathcal{S}_{c+n-r}}(t)}{1 - t^n}.$$

Proof. Let $c' := c + n - 2r = nq + n - r$ and $r' := n - r$. Since $c' + n - 2r' = c$, Theorem 2.7 yields

$$\begin{aligned} H_{\mathcal{S}_c}(t) &= t^r H_{\mathcal{S}_{c-r}}(t) + H_{\mathcal{S}_{c+n-r}}(t) - t^r H_{\mathcal{S}_{c'}}(t) \quad \text{and} \\ H_{\mathcal{S}_{c'}}(t) &= t^{r'} H_{\mathcal{S}_{c-r}}(t) + H_{\mathcal{S}_{c+n-r}}(t) - t^{r'} H_{\mathcal{S}_c}(t). \end{aligned}$$

Substituting $H_{\mathcal{S}_{c'}}(t)$ in the first formula and solving for $H_{\mathcal{S}_c}(t)$ gives the result. \square

Turning back to the question of computing the Hilbert-series of the R -modules $M_r = \text{Syz}_R(X^r, Y^r, Z^r)$, $1 \leq r \leq n-1$, we obtain via Theorem 2.8

$$\begin{aligned}
 H_{M_r}(t) &= \frac{(t^r - t^n) \cdot H_{\text{Syz}_R(1, Y^r, Z^r)}(t) + (1 - t^r) H_{\text{Syz}_R(-Y^n - Z^n, Y^r, Z^r)}(t)}{1 - t^n} \\
 &= \frac{2t^r(t^r - t^n) + (1 - t^r)(t^n + t^{2r})}{(1 - t)^3} \\
 (2.1) \quad &= \frac{t^n + 3t^{2r} - 3t^{n+r} - t^{3r}}{(1 - t)^3}.
 \end{aligned}$$

Since the Hilbert-series of M_r and M_s for $1 \leq r < s \leq n-1$ are not multiples of each other, the modules $(M_r)_{1 \leq r \leq n-1}$ are pairwise non-isomorphic.

Now we are able to prove the following.

Theorem 2.9. *Assume we are in Situation 2.1. The bundle $\text{Syz}_C(X^a, Y^a, Z^a)$ admits a twisted Frobenius periodicity if and only if $\delta\left(\frac{a}{n}, \frac{a}{n}, \frac{a}{n}\right) = 0$. Moreover, the length of this periodicity is bounded from above by the order of p in $\mathbb{Z}/(2n)$.*

Proof. This follows from Corollary 1.5 and because the isomorphism class of the module $\text{Syz}_R(X^{aq}, Y^{aq}, Z^{aq})$ depends on $r \equiv aq \pmod{n}$ and the parity of $\frac{aq-r}{n}$. \square

Remark 2.10. One should mention that one already knew that strongly semistable bundles admit a twisted Frobenius periodicity. This is because the coefficients of the equation $X^n + Y^n + Z^n = 0$ lie in a finite field and therefore all moduli spaces are finite dimensional varieties over \mathbb{F}_p . Hence, the number of \mathbb{F}_p -rational points gives an upper bound for the length of the periodicity but it's very rough and there is no hint how to compute it explicitly.

The next example shows that the upper bound for the length of the periodicity from Theorem 2.9 is the best possible.

Example 2.11. Let $p = 37$ and $n = 14$. Then $p^e = 14 \cdot \theta + r$ with

$$(\theta, r) = \begin{cases} (\text{even}, 9) & \text{if } e \equiv 1 \pmod{3}, \\ (\text{odd}, 11) & \text{if } e \equiv 2 \pmod{3}, \\ (\text{even}, 1) & \text{if } e \equiv 0 \pmod{3}. \end{cases}$$

From this computation it is easy to see that $\delta\left(\frac{1}{14}, \frac{1}{14}, \frac{1}{14}\right) = 0$, hence $\Omega_{\mathbb{P}^2|C}$ is strongly semistable and we get a twisted $(0, 3)$ -Frobenius periodicity

$$F^{3*}(\Omega_{\mathbb{P}^2|C}) \cong \Omega_{\mathbb{P}^2|C} \left(-\frac{3}{2} \cdot (q-1)\right).$$

Theorem 2.9 enables us to compute the Hilbert-Kunz function of (X^a, Y^a, Z^a) , if the bundle $\text{Syz}_C(X^a, Y^a, Z^a)$ is strongly semistable and none of its Frobenius pull-backs split as a direct sum of twisted copies of the structure sheaf.

Theorem 2.12. *Assume we are in Situation 2.1 and that $\text{Syz}_C(I)$ is strongly semistable and none of its Frobenius pull-backs are a direct sum of twisted copies of the structure sheaf. Let $aq = ap^e = n\theta + r$ with $\theta \in \mathbb{N}$ and $0 \leq r < n$. Then*

$$\text{HK}(I, q) = \begin{cases} \frac{3a^2n}{4} \cdot q^2 - \frac{3n}{4}r^2 + r^3 & \text{if } \theta \text{ is even,} \\ \frac{3a^2n}{4} \cdot q^2 - \frac{3n}{4}(n-r)^2 + (n-r)^3 & \text{if } \theta \text{ is odd.} \end{cases}$$

Proof. Under our assumptions on $\text{Syz}_C(I)$, the quotients $R/(X^{aq}, Y^{aq}, Z^{aq})$ have infinite projective dimension and their resolutions are given by Theorem 2.5. By Corollary 2.6 we have an isomorphism $\text{Syz}_R(I^{[q]}) \cong \text{Syz}_R(J)(l)$, where depending on θ , we use J and b to denote either (X^r, Y^r, Z^r) and r or $(X^{n-r}, Y^{n-r}, Z^{n-r})$ and $n-r$. We obtain a diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Syz}_C(I^{[q]})(m) & \longrightarrow & \mathcal{O}_C(m - ap^e)^3 & \longrightarrow & \mathcal{O}_C(m) \longrightarrow 0 \\ & & \downarrow \cong & & & & \\ 0 & \longrightarrow & \text{Syz}_C(J)(m+l) & \longrightarrow & \mathcal{O}_C(m+l-b)^3 & \longrightarrow & \mathcal{O}_C(m+l) \longrightarrow 0. \end{array}$$

Taking global sections as in Equation (1.1) the claim follows by using

$$h^0\left(C, \text{Syz}_C\left(X^{ap^e}, Y^{ap^e}, Z^{ap^e}\right)(m)\right) = h^0\left(C, \text{Syz}_C\left(X^b, Y^b, Z^b\right)(m+l)\right)$$

and a computation similar to that in the proof of [BK13, Corollary 4.1]. \square

Example 2.13. Let $p = 37$ and $n = 14$. By Example 2.11 we obtain the Hilbert-Kunz function

$$\text{HK}(R, 37^e) = \begin{cases} \frac{21}{2} \cdot 37^{2e} - \frac{243}{2} & \text{if } e \equiv 1 \pmod{3}, \\ \frac{21}{2} \cdot 37^{2e} - \frac{135}{2} & \text{if } e \equiv 2 \pmod{3}, \\ \frac{21}{2} \cdot 37^{2e} - \frac{19}{2} & \text{if } e \equiv 0 \pmod{3}. \end{cases}$$

In [BK13] the authors proved that $\Omega_{\mathbb{P}^2|_C}$ admits a twisted $(0,1)$ -Frobenius periodicity if $p \equiv -1 \pmod{2n}$. The authors tried to adopt their proof to the case $p \equiv 1 \pmod{2n}$ but failed (cf. [BK13, Remark 3.5]).

Theorem 2.14. *Assume $p \equiv \pm 1 \pmod{2n}$ in Situation 2.1. Then $\Omega_{\mathbb{P}^2|_C}$ is strongly semistable with twisted $(0,1)$ -Frobenius periodicity*

$$F^{e*}(\Omega_{\mathbb{P}^2|_C}) \cong \Omega_{\mathbb{P}^2|_C} \left(-\frac{3}{2} \cdot (p^e - 1) \right).$$

Moreover, the Hilbert-Kunz function of R is given by

$$\text{HK}(R, p^e) = \frac{3n}{4} \cdot p^{2e} + 1 - \frac{3n}{4}.$$

Proof. As $\delta(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}) = 0$, the bundle $\Omega_{\mathbb{P}^2|_C}$ is strongly semistable. Since p has to be odd by assumption and $a = 1$, the Frobenius pull-backs of $\Omega_{\mathbb{P}^2|_C}$ are not of the form $\mathcal{O}_C(l)^2$ as mentioned at the beginning of this section. If $p \equiv 1 \pmod{2n}$ then all powers p^e can be written in the form $\text{even} \cdot n + 1$ and if $p \equiv -1 \pmod{2n}$ all powers p^{2e} can be written as $\text{even} \cdot n + 1$ and the powers p^{2e+1} can be written as $\text{odd} \cdot n + n - 1$. The periodicity of $\text{Syz}_C(X, Y, Z)$ follows from Corollary 2.6 and the statement about the Hilbert-Kunz function is due to Theorem 2.12. \square

Remark 2.15. Note that one can construct (s, t) -Frobenius periodicities in the classical sense, e.g. of degree zero bundles, from the twisted (s, t) -Frobenius periodicities obtained from Theorem 2.9 as follows:

If a is even, one can consider the bundle $\text{Syz}_C(X^a, Y^a, Z^a)(\frac{3a}{2})$.

If a is odd, one obtains a (s, t) -Frobenius periodicity of $\text{Syz}_D(U^{2a}, V^{2a}, W^{2a})(3a)$ on the Fermat curve D of degree $2n$ as it was done in [BK13, Example 5.1].

Recall that due to Lange and Stuhler the vector bundles of degree zero admitting a $(0, t)$ -Frobenius periodicity are exactly those that are étale trivializable (cf. [LS77]). For the syzygy bundles which only admit a (s, t) -Frobenius periodicity with $s \geq 1$ there is only a finite trivialization, namely the composition of the Frobenius morphism and a suitable étale covering.

By Theorem 2.14 and this remark, the syzygy bundle $\text{Syz}_D(U^2, V^2, W^2)(3)$ admits a $(0, 1)$ -Frobenius periodicity on the projective Fermat curve D of degree $2n$, provided $p \equiv \pm 1$ modulo $2n$. Stäbler computed the étale map trivializing this bundle in characteristics $p \equiv -1$ modulo $2n$ explicitly in [Stä11].

3. WHICH FROBENIUS PERIODICITIES CAN BE ACHIEVED?

The next examples deal with the question which twisted $(0, t)$ -Frobenius periodicities of the bundles $\Omega_{\mathbb{P}^2|C}$ can be achieved. A sufficient condition for having a twisted Frobenius periodicity is (cf. Theorem 2.9) $\delta(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}) = 0$, which is equivalent to the condition that the distances of all triples $v_e := (\frac{p^e}{n}, \frac{p^e}{n}, \frac{p^e}{n})$ to L_{odd} are at least one. Let $p^e = \theta_e \cdot n + r_e$ with $\theta_e, r_e \in \mathbb{N}$ and $0 \leq r_e < n$. The nearest element to v_e in L_{odd} , which potentially has a taxicab distance < 1 from v_e , is given by the component-wise rounding ups of v_e if θ_e is even and by the component-wise rounding downs of v_e if θ_e is odd. This leads to the (sufficient) conditions

$$\left. \begin{array}{ll} 3 \cdot (1 - \frac{r_e}{n}) \geq 1 & \text{if } \theta_e \text{ is even,} \\ 3 \cdot \frac{r_e}{n} \geq 1 & \text{if } \theta_e \text{ is odd.} \end{array} \right\} \iff \left\{ \begin{array}{ll} 2 \cdot n \geq 3 \cdot r_e & \text{if } \theta_e \text{ is even,} \\ 3 \cdot r_e \geq n & \text{if } \theta_e \text{ is odd.} \end{array} \right.$$

Example 3.1. Let p be odd, $l \in \mathbb{N}$ and $n = \frac{p^{l+1}+1}{2}$. For $0 \leq e \leq 2l+2$ we have

$$(3.1) \quad p^e = \begin{cases} 0 \cdot n + p^e & \text{if } 0 \leq e \leq l, \\ (2 \cdot p^{e-l-1} - 1) \cdot n + n - p^{e-l-1} & \text{if } l+1 \leq e \leq 2l+1, \\ (2 \cdot p^{l+1} - 2) \cdot n + 1 & \text{if } e = 2l+2. \end{cases}$$

This shows that p^e is of the form $\text{even} \cdot n + p^{e'}$ or $\text{odd} \cdot n + n - p^{e'}$ for some $0 \leq e' \leq l$. Since $2n = p^{l+1} + 1 \geq 3p^{e'}$ for all $0 \leq e' \leq l$, we see that $\delta(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}) = 0$. By Corollary 2.6 we find that $F^{e*}(\Omega_{\mathbb{P}^2|C})$ is isomorphic to

$$F^{e'*}(\Omega_{\mathbb{P}^2|C}) \left(-\frac{3}{2} \cdot (p^e - p^{e'}) \right),$$

where $e' \equiv e \pmod{l+1}$ with $e' \in \{0, \dots, l\}$. With Equation (2.1) we find a twisted $(0, l+1)$ -Frobenius periodicity. The Hilbert-Kunz function is given by

$$\text{HK}(R, p^e) = \frac{3n}{4} \cdot (p^{2e} - p^{2e'}) + p^{3e'},$$

where again $e' \equiv e \pmod{l+1}$ with $e' \in \{0, \dots, l\}$.

Example 3.2. Let $p = 2$, $1 \leq l \in \mathbb{N}$ and n odd with $2^l < n < 2^{l+1}$, hence $n = 2^l + x$ with $0 < x < 2^l$. We want to show that $\Omega_{\mathbb{P}^2|C}$ admits a twisted Frobenius periodicity if and only if $n = 3$. The condition $\delta(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}) = 0$ forces

$$2n = 2^{l+1} + 2x \geq 3 \cdot 2^l \geq 3 \cdot 2^e$$

for all $0 \leq e \leq l$. This is equivalent to $x \geq 2^{l-1}$. Since $2^{l+1} = n + 2^l - x$ with $0 < 2^l - x < 2^l < n$, we need that the inequality $3 \cdot (2^l - x) \geq n$ holds. This is equivalent to $x \leq 2^{l-1}$. This shows that a twisted Frobenius periodicity might

appear only in the case $x = 2^{l-1}$ resp. $n = 3 \cdot 2^{l-1}$. Since n is odd, we obtain the single possibility $n = 3$. Now let $n = 3$. Then $\Omega_{\mathbb{P}^2}|_C$ is strongly semistable since $\delta(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 0$. We remark that this is well-known: since C is an elliptic curve and $\Omega_{\mathbb{P}^2}|_C$ is semistable the strong semistability follows also from [MR82, Theorem 2.1].

4. THE NON-STRONGLY SEMISTABLE CASE

In this section we want to compute the Hilbert-Kunz function of $I = (X^a, Y^a, Z^a)$ under the condition that the syzygy bundle $\text{Syz}_C(X^a, Y^a, Z^a)$ is *not* strongly semistable on the Fermat curve C . We do this by an explicit computation of the strong HN-filtration of these bundles.

Lemma 4.1. *In addition to Situation 2.1 let b denote a natural number and write $b = nl + r$ with $0 \leq r \leq n - 1$.*

- (1) *If l is even and $r > \frac{2n}{3}$, then $\text{Syz}_C(X^b, Y^b, Z^b)$ is not semistable on C . Moreover, one has*

$$\Gamma(C, \text{Syz}_C(X^b, Y^b, Z^b)(n(l+1 + \frac{l}{2}))) \neq 0.$$

- (2) *If l is odd and $r < \frac{n}{3}$, then $\text{Syz}_C(X^b, Y^b, Z^b)$ is not semistable on C . Moreover, one has*

$$\Gamma(C, \text{Syz}_C(X^b, Y^b, Z^b)(n(l + \lfloor \frac{l}{2} \rfloor) + 3r)) \neq 0.$$

Proof. For the proof of (1) see [Bre05a, Proposition 1]. For the proof of part (2) which is rather similar to (1) see [Kai09, Lemma 4.2.8(2)]. \square

We list the following two neat consequences of the previous lemma.

Corollary 4.2. *In Situation 2.1 assume $p \equiv n \pm 1 \pmod{2n}$ for an even natural number $n \geq 4$. Then $F^*(\Omega_{\mathbb{P}^2}|_C)$ is not semistable on C .*

Proof. For $p \equiv n + 1 \pmod{2n}$ this follows from part (1) of Lemma 4.1 and for the case $p \equiv n - 1 \pmod{2n}$ one applies part (2). \square

Corollary 4.3. *If in Situation 2.1 we have $a \equiv n \pmod{2n}$, then $\text{Syz}_C(X^a, Y^a, Z^a)$ is not semistable on C .*

Proof. This follows immediately from part (2) of Lemma 4.1. \square

Now, using Lemma 4.1 we are able to prove that the s th Frobenius pull-back (where s is the integer in Han's Theorem 1.3) is not semistable. Moreover, we explicitly compute a strong Harder-Narasimhan filtration of $\text{Syz}_C(X^a, Y^a, Z^a)$.

Theorem 4.4. *In Situation 2.1 assume $p > \frac{3a}{2n}$. If $\delta(\frac{a}{n}, \frac{a}{n}, \frac{a}{n}) \neq 0$ and s is the integer of Han's Theorem 1.3, then $F^{s*}(\text{Syz}_C(X^a, Y^a, Z^a))$ is not semistable. Let $ap^s = nl + r$ with $l \in \mathbb{N}$ and $0 \leq r \leq n - 1$. Then the Frobenius pull-back $F^{s*}(\text{Syz}_C(X^a, Y^a, Z^a))$ has a strong Harder-Narasimhan filtration given by*

$$0 \rightarrow \mathcal{O}_C(-m) \rightarrow F^{s*}(\text{Syz}_C(X^a, Y^a, Z^a)) \rightarrow \mathcal{O}_C(m - 3ap^s) \rightarrow 0,$$

with $m = n(l+1+\frac{l}{2})$ if l is even and $m = n(l+\lfloor \frac{l}{2} \rfloor) + 3r$ if l is odd. Moreover, this filtration is minimal for $p \geq n-3$ and $s \geq 1$ in the sense that $F^{(s-1)}(\text{Syz}_C(X^a, Y^a, Z^a))$ is semistable.*

Proof. Since $\mathcal{S} := \text{Syz}_C(X^a, Y^a, Z^a)$ is not strongly semistable, we know by Han's Theorem 1.3 and Corollary 1.5 that the taxicab distance from $(\frac{ap^s}{n}, \frac{ap^s}{n}, \frac{ap^s}{n})$ to the nearest element in L_{odd} is < 1 . We have $s \geq 0$ due to the assumption $p > \frac{3a}{2n}$. We compute the taxicab distance in dependence on l . First, we consider the case where l is even. So the distance from $\frac{ap^s}{n}$ to the nearest odd integer is $\frac{n-r}{n}$ and the taxicab distance to the closest element in L_{odd} is $3\frac{n-r}{n} = 3 - \frac{3r}{n}$, which is by assumption < 1 . Hence, we obtain $r > \frac{2n}{3}$. Now we apply Lemma 4.1(1) to $b = ap^s$ and see that $F^{s*}(\mathcal{S}) \cong \text{Syz}_C(X^{ap^s}, Y^{ap^s}, Z^{ap^s})$ is not semistable. Moreover, we have a non-trivial mapping $\mathcal{O}_C(-n(l+1+\frac{l}{2})) \rightarrow F^{s*}(\mathcal{S})$. We want to prove that this mapping constitutes the HN-filtration of the s th Frobenius pull-back, i.e., the mapping has no zeros on C . First, we compute the Hilbert-Kunz multiplicity of the ideal $I := (X^a, Y^a, Z^a)$ in the ring R . Since

$$\delta\left(\frac{a}{n}, \frac{a}{n}, \frac{a}{n}\right) = \frac{1}{p^s} \left(1 - \left(3 - \frac{3r}{n}\right)\right) = \frac{1}{p^s} \left(\frac{3r-2n}{n}\right)$$

we obtain

$$e_{\text{HK}}(I) = \frac{3n}{4}a^2 + \frac{n^3}{4} \left(\frac{1}{p^s} \left(\frac{3r-2n}{n}\right)\right)^2 = \frac{3n}{4}a^2 + \frac{(3r-2n)^2}{4p^{2s}} \cdot n$$

via Theorem 1.4. Now we use the Hilbert-Kunz multiplicity $e_{\text{HK}}(I)$ to read off the degree of the destabilizing invertible sheaf $\mathcal{L} \subset F^{s*}(\mathcal{S})$. Theorem 1.1(2) yields

$$\deg(\mathcal{L}) = -np^s \left(\frac{3a}{2} - \sqrt{\frac{(3r-2n)^2}{4p^{2s}}}\right) = -n^2(l+1+\frac{l}{2}).$$

Thus $\deg(\mathcal{L} \otimes \mathcal{O}_C(n(l+1+\frac{l}{2}))) = 0$. Since this line bundle does have a non-trivial section, we obtain $\mathcal{L} \cong \mathcal{O}_C(-n(l+1+\frac{l}{2}))$ and the HN-filtration is indeed

$$0 \subset \mathcal{O}_C(-n(l+1+\frac{l}{2})) \subset F^{s*}(\mathcal{S}).$$

The assertion on the quotient line bundle is clear since the determinant bundle is additive on short exact sequences.

The case that l is odd follows essentially in the same way and we omit it here. Finally, we prove the assertion about the minimality of the HN-filtration. Assume the minimal integer e such that $F^{e*}(\mathcal{S})$ is not semistable is strictly smaller than s . If l is even, we obtain the equality $\ell = n(3r-2n)p^{e-s}$, where the integer ℓ is defined as in Theorem 1.1(4). But this equality can only hold for prime numbers $p \leq n-3$ since $0 < 3r-2n \leq n-3$ (we have $r \leq n-1$ and $p \nmid n$).

If l is odd, we have $\ell = n(n-3r)p^{e-s}$. If $r \geq 1$, we can conclude as above. In case of $r = 0$, we see that p^s has to divide l and we can conclude that $a = nb$ with b odd. But this means that the minimal s of Han's Theorem is actually $s = 0$ which contradicts the assumption $s \geq 1$. \square

If the characteristic is sufficiently large, Theorem 4.4 also yields a numerical criterion for semistability of the syzygy bundle $\text{Syz}_C(X^a, Y^a, Z^a)$. Semistability of these bundles in characteristic 0 is part of Section 5.

Corollary 4.5. *Assume the situation of Theorem 4.4. If $\text{char}(k) \geq n-3$, then $\text{Syz}_C(X^a, Y^a, Z^a)$ is semistable if and only if $s > 0$.*

Proof. This is an immediate consequence of Theorem 4.4. \square

Example 4.6. Let $p \neq 3$ and consider the Fermat cubic C ($n = 3$), which is an elliptic curve. We recall that by [MR82, Theorem 2.1] the syzygy bundle $\text{Syz}_C(X^a, Y^a, Z^a)$ is semistable on C if and only if it is strongly semistable, which is by Corollary 1.5 equivalent to $\delta\left(\frac{a}{n}, \frac{a}{n}, \frac{a}{n}\right) = 0$. If $p > \frac{a}{2}$, it follows from Theorem 4.4 that this bundle is not semistable on C if and only if $a = 3l$ for some odd integer l . This description for non-semistability does not hold when $p \leq \frac{a}{2}$. For instance, let $p = 2$ and $a = 3 \cdot 2 + 1 = 7$. Then the taxicab distance from $(\frac{7}{6}, \frac{7}{6}, \frac{7}{6})$ to $(1, 1, 1) \in L_{\text{odd}}$ equals $\frac{3}{6} < 1$ and hence $\delta\left(\frac{7}{3}, \frac{7}{3}, \frac{7}{3}\right) \neq 0$.

Combining Theorem 4.4 with [Mil00, Theorem 2.1], one obtains the Hilbert-Kunz functions of the ideals (X^a, Y^a, Z^a) for $q = p^e \gg 0$. Note that the term $e_{\text{HK}}(I)$ can be explicitly computed via Theorem 1.4.

Corollary 4.7. *In the situation of Theorem 4.4 one has*

$$(4.1) \quad \text{HK}(I, p^e) = e_{\text{HK}}(I)p^{2e} \text{ for all } e \gg 0.$$

Example 4.8. Let $p = 3$ and $n = 7$. Then $\delta\left(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right) = \frac{1}{63}$ with $s = 2$. By Theorem 4.4 we have the strong Harder-Narasimhan filtration

$$0 \rightarrow \mathcal{O}_C(-13) \rightarrow F^{2*}(\text{Syz}_C(X, Y, Z)) \rightarrow \mathcal{O}_C(-14) \rightarrow 0.$$

Since $-3^t + 4 < 0 \Leftrightarrow t \geq 2$, the e -th Frobenius pull-backs of $\text{Syz}_C(X, Y, Z)$ split as

$$\mathcal{O}_C(-13 \cdot p^{e-2}) \oplus \mathcal{O}_C(-14 \cdot p^{e-2})$$

for $e \geq 4$. This gives

$$\text{HK}(R, 3^e) = e_{\text{HK}}(R) \cdot 3^e = \frac{427}{81} \cdot 3^e$$

for $e \geq 4$. The other values are $\text{HK}(R, 3^0) = 1$, $\text{HK}(R, 3^1) = 27$, $\text{HK}(R, 3^2) = 419$ and $\text{HK}(R, 3^3) = 3843$ by an explicit computation with CoCoA [CoC]. Moreover, another explicit computation shows

$$\begin{aligned} \text{Syz}_R(X^9, Y^9, Z^9) &\cong \text{Syz}_R(X^5, Y^5, Z^5)(-6) \text{ and} \\ \text{Syz}_R(X^{27}, Y^{27}, Z^{27}) &\cong R(-39) \oplus R(-42). \end{aligned}$$

Remark 4.9. We want to relate our results on the Hilbert-Kunz functions of the ideals (X^a, Y^a, Z^a) with those of other authors. As a consequence of a result of Brenner [Bre07, Theorem 6.1], the Hilbert-Kunz function of an (X, Y, Z) -primary ideal I of the ring $R = k[X, Y, Z]/(X^n + Y^n + Z^n)$ has the form

$$(4.2) \quad e \mapsto e_{\text{HK}}(I) \cdot p^{2e} + \phi(p, e),$$

where $\phi(p, _)$ is an eventually periodic function. By Equation (4.1), we see that $\phi(p, e) = 0$ for all large e and all p coprime to n if some Frobenius pull-back of $\text{Syz}_C(I)$ splits as a direct sum of twisted structure sheaves. If this is not the case, we obtain from Theorem 2.12 that the shape of $\phi(p, _)$ does only depend on the residue class of p modulo $2n$. All in all, we have seen that for every fixed n there are only finitely many possibilities for $\phi(p, _)$.

5. STRONGLY SEMISTABLE REDUCTION ON THE RELATIVE FERMAT CURVE

In this section we deal with a problem proposed by Brenner in [Bre05b, Problem 5] which contains a special case of Miyaoka's problem [Miy87, Problem 5.4].

Problem 5.1 (Brenner). How does the strong semistability of $\text{Syz}_C(X^a, Y^a, Z^a)$ on the Fermat curve C of degree n depend on the characteristic p , the degree n and the integer a ? In particular, for fixed n and a , is the set of prime numbers such that $\text{Syz}_C(X^a, Y^a, Z^a)$ is strongly semistable finite, infinite or does it contain almost all prime numbers?

We have already answered the first part by Theorem 1.4 and Corollary 1.5 in Section 1 (cf. also Example 1.6). The following theorem gives a numerical criterion for semistability of the syzygy bundle $\text{Syz}_C(X^a, Y^a, Z^a)$ on a Fermat curve in characteristic 0. It also shows that if the syzygy bundle is semistable in characteristic 0, then it has strongly semistable reduction for infinitely many prime numbers. Before we state the theorem, we recall some notation for a relative curve $\mathcal{C} \rightarrow \text{Spec } \mathbb{Z}$. For a prime number p we denote by $\mathcal{C}_p := \mathcal{C} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{F}_p$ the *special fiber* over the closed point $(p) \in \text{Spec } \mathbb{Z}$ and by $\mathcal{C}_0 := \mathcal{C} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Q}$ the *generic fiber* over the generic point $(0) \in \text{Spec } \mathbb{Z}$. Finally, we recall that by a theorem of Dirichlet there exist infinitely many prime numbers in any arithmetic progression.

Theorem 5.2. *Let $a \geq 1$ be an integer, and consider the smooth projective relative Fermat curve $\mathcal{C} := \text{Proj}(\mathbb{Z}_n[X, Y, Z]/(X^n + Y^n + Z^n)) \rightarrow \text{Spec } \mathbb{Z}_n$. Write $a = nl + r$ with $0 \leq r < n$ and let $\tilde{r} \equiv a \pmod{2n}$ ($0 \leq \tilde{r} < 2n$). Then the following conditions are equivalent:*

- (1) *One has $r \leq \frac{2n}{3}$ if l is even and $r \geq \frac{n}{3}$ if l is odd.*
- (2) *One has $\tilde{r} \leq \frac{2n}{3}$ if $\tilde{r} < n$ and $\tilde{r} \geq \frac{4n}{3}$ if $\tilde{r} \geq n$.*
- (3) *For all prime numbers $p > \max\{n - 3, \frac{3a}{2n}\}$ the integer s of Han's Theorem 1.3 is either ≥ 1 or $\delta(\frac{a}{n}, \frac{a}{n}, \frac{a}{n}) = 0$.*
- (4) *The syzygy bundle $\text{Syz}_{\mathcal{C}_p}(X^a, Y^a, Z^a)$ is strongly semistable on the special fiber \mathcal{C}_p for all prime numbers $p \equiv \pm 1 \pmod{2n}$ with $p > \frac{3a}{2n}$.*
- (5) *The syzygy bundle $\text{Syz}_{\mathcal{C}_0}(X^a, Y^a, Z^a)$ is semistable on the generic fiber \mathcal{C}_0 .*

Proof. (1) \Leftrightarrow (2). This is obvious.

(1) \Rightarrow (3). Assume there is a prime number $p > \frac{3a}{2n}$ such that $s = 0$. Then $\text{Syz}_{\mathcal{C}_p}(X^a, Y^a, Z^a)$ is not semistable on the fiber \mathcal{C}_p by Theorem 4.4. The proof of Theorem 4.4 shows that we have either $r > \frac{2n}{3}$ or $r < \frac{n}{3}$ depending on l . But this contradicts the assumption.

(3) \Rightarrow (1). Suppose the assumption on r in (1) does not hold. Then the bundle $\text{Syz}_{\mathcal{C}_p}(X^a, Y^a, Z^a)$ is non-semistable on every fiber \mathcal{C}_p by Lemma 4.1. But this contradicts Theorem 4.4 since $s \geq 1$ and $F^{s*}(\text{Syz}_{\mathcal{C}_p}(X^a, Y^a, Z^a))$ is the first non-semistable Frobenius pull-back in characteristics $p \geq n - 3$.

(2) \Rightarrow (4). We have $p^s a \equiv \pm \tilde{r} \pmod{2n}$ for every prime number $p \equiv \pm 1 \pmod{2n}$ and every $s \geq 0$. If $\tilde{r} < n$, then the distance from $\frac{p^s}{n}$ to the nearest odd integer is $\frac{n - \tilde{r}}{n}$. Hence the taxicab distance from $(\frac{p^s a}{n}, \frac{p^s a}{n}, \frac{p^s a}{n})$ to the nearest element in L_{odd} is

$$3 \frac{n - \tilde{r}}{n} = 3 - \frac{3\tilde{r}}{n} \geq 3 - \frac{2n}{n} = 1.$$

Similarly, if $\tilde{r} \geq n$, then the distance from $\frac{p^s a}{n}$ to the nearest odd integer is $\frac{\tilde{r}-n}{n}$, and thus the taxicab distance to the nearest element in L_{odd} is

$$3 \frac{\tilde{r} - n}{n} = \frac{3\tilde{r}}{n} - 3 \geq \frac{4n}{n} - 3 = 1.$$

Hence $\delta(\frac{a}{n}, \frac{a}{n}, \frac{a}{n}) = 0$ in characteristics $p \equiv \pm 1 \pmod{2n}$ (with $p > \frac{3a}{2n}$) and the syzygy bundle $\text{Syz}_{C_p}(X^a, Y^a, Z^a)$ is strongly semistable by Corollary 1.5.

(4) \Rightarrow (5). This follows from the openness of semistability (see [Miy87, paragraph after Proposition 5.2]).

(5) \Rightarrow (1). This follows immediately from Lemma 4.1. \square

Remark 5.3. To determine the semistability of the bundles $\text{Syz}_C(X^a, Y^a, Z^a)$ one might also consider the application of restriction theorems. It is well-known that the vector bundle $\mathcal{S} := \text{Syz}_{\mathbb{P}^2}(X^a, Y^a, Z^a)$ is stable on the projective plane with Chern classes $c_1(\mathcal{S}) = -3a$ and $c_2(\mathcal{S}) = 3a^2$ (see for instance [Bre08a, Corollary 3.2]). Hence its discriminant equals $\Delta(\mathcal{S}) = 4c_2(\mathcal{S}) - c_1(\mathcal{S})^2 = 3a^2$. So the restriction theorem of Langer [Lan09, Theorem 2.19] tells us that $\mathcal{S}|_C$ is stable on every smooth curve C of degree $n > \frac{3a^2+1}{2}$. But this bound grows quadratically with a and therefore becomes expeditiously high.

It was shown in [Bre05a, Corollary 2] that the set of primes where the bundle $\text{Syz}_C(X^a, Y^a, Z^a)$ has strongly semistable reduction contains in general not almost all prime numbers disproving a stronger version of [Miy87, Problem 5.4] which was conjectured by N. I. Shepherd-Barron in [SB97]. That is, strong semistability is not an open property in arithmetic deformations. The following example shows that this phenomenon depends on the pair (a, n) . But using Han's Theorem 1.3 and Corollary 1.5 one can exhibit also this property explicitly.

Example 5.4. Let $n = 5$ and p be an odd prime number with $\gcd(p, 5) = 1$. For all $e \in \mathbb{N}$ we can write $p^e = 10 \cdot l + r$ for some $l \in \mathbb{N}$ and $r \in \{1, 3, 7, 9\}$. Then $\frac{p^e}{5}$ is $2l + \frac{1}{5}$, $2l + \frac{3}{5}$, $2l + 1 + \frac{2}{5}$ resp. $2l + 1 + \frac{4}{5}$ and hence the nearest element in L_{odd} to $(\frac{p^e}{5}, \frac{p^e}{5}, \frac{p^e}{5})$ is given by $(2l + 1, 2l, 2l)$, $(2l + 1, 2l + 1, 2l + 1)$, $(2l + 1, 2l + 1, 2l + 1)$ resp. $(2l + 1, 2l + 2, 2l + 2)$. Hence, the taxicab distance of $(\frac{p^e}{5}, \frac{p^e}{5}, \frac{p^e}{5})$ to L_{odd} is $\frac{6}{5}$ in any case, showing $\delta(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}) = 0$. Therefore, $\Omega_{\mathbb{P}^2}|_C$ is strongly semistable in almost all characteristics and semistable in characteristic zero by Theorem 5.2. The only exceptional prime numbers are 2 and 5.

If $\text{Syz}(X^a, Y^a, Z^a)$ is not semistable on the Fermat curve in characteristic 0 then Theorem 5.2 allows to compute its Harder-Narasimhan filtration via reduction to positive characteristic. It turns out that the HN-filtration in characteristic 0 coincides with the HN-filtration in positive characteristic p (see Theorem 4.4) for almost all prime numbers.

Theorem 5.5. *Let k be a field of characteristic 0 and denote by*

$$C := \text{Proj}(k[X, Y, Z]/(X^n + Y^n + Z^n))$$

the Fermat curve of degree n over k . Further, let $a \geq 1$ be an integer and write $a = nl + r$ with $0 \leq r < n$. Set $\mathcal{S} := \text{Syz}_C(X^a, Y^a, Z^a)$. If l is even and $r > \frac{2n}{3}$ or l is odd and $r < \frac{n}{3}$, then \mathcal{S} is not semistable and its HN-filtration is given by the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(-m) \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_C(m - 3a) \longrightarrow 0,$$

where m is defined as in Theorem 4.4 in dependence of l .

Proof. By Lemma 4.1 we have a non-trivial mapping $\mathcal{O}_C(-m) \rightarrow \mathcal{S}$. We have to show that this mapping has no zeros on C . Since C is already defined over \mathbb{Z} we can reduce to the case $k = \mathbb{Q}$. Assume that we have a factorization

$$\mathcal{O}_C(-m) \rightarrow \mathcal{L} \rightarrow \text{Syz}_C(X^a, Y^a, Z^a),$$

where \mathcal{L} is the maximal destabilizing subbundle. As in Theorem 5.2 we consider C as the generic fiber of the relative curve

$$C = \text{Proj}(\mathbb{Z}_n[X, Y, Z]/(X^n + Y^n + Z^n)) \rightarrow \text{Spec } \mathbb{Z}_n.$$

On every special fiber C_p satisfying $p > \max\{n - 3, \frac{3a}{2n}\}$ the HN-filtration of the restriction equals $0 \subset \mathcal{O}_{C_p}(-m) \subset \text{Syz}_{C_p}(X^a, Y^a, Z^a)$ by Theorem 4.4. Since the HN-filtration of $\text{Syz}_C(X^a, Y^a, Z^a)$ extends to an open subset of $\text{Spec } \mathbb{Z}_n$, we obtain $\mathcal{L} \cong \mathcal{O}_C(-m)$. \square

Example 5.6. In this example we provide an application of our results to the theory of *tight closure* which is related to Hilbert-Kunz theory and strongly connected to the theory of (strongly) semistable vector bundles due to the work of Brenner (see [Bre08b]). Via this geometric approach Problem 5.1 is related to Hochster's question [Hoc94, Question 13]. In the homogeneous coordinate ring of the Fermat septic ($n = 7$), Brenner and Katzman have shown in [BK06, Theorem 4.1] by tedious computations that $X^3Y^3 \in (X^4, Y^4, Z^4)^*$ (the tight closure) for prime numbers $p \equiv 3 \pmod{7}$ and $X^3Y^3 \notin (X^4, Y^4, Z^4)^*$ for $p \equiv 2 \pmod{7}$. That is, the relative curve $C : \text{Proj}(\mathbb{Z}[X, Y, Z]/(X^7 + Y^7 + Z^7)) \rightarrow \text{Spec } \mathbb{Z}$ illustrates that tight closure does not behave uniformly in the fibers of an arithmetic deformation. A similar reasoning as in Examples 1.6 and 5.4 shows that $\text{Syz}_C(X^4, Y^4, Z^4)$ is strongly semistable on the Fermat septic if and only if $p \equiv \pm 1 \pmod{7}$. In particular, the monomial X^3Y^3 of degree 6 belongs to the tight closure of the ideal (X^4, Y^4, Z^4) in $k[X, Y, Z]/(X^7 + Y^7 + Z^7)$ in these characteristics by [Bre08b, Theorem 6.4]. Hence, we easily get infinitely many prime numbers where the tight closure inclusion does hold. Furthermore, the question for which prime numbers the syzygy bundle $\text{Syz}(X^4, Y^4, Z^4)$ has strongly semistable reduction, raised in [BK06, paragraph before Corollary 4.3], is now answered.

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